

NPS-53WG72113A

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ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

by

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20 November 1972

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ABSTRACT:

A complete asymptotic development of the Stirling numbers $S(N,K)$ of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans. Roy. Soc. Canad., 49(1955) 49-54] and de Bruijn [Asymptotic Methods in Analysis, North-Holland Publishing Co., Amsterdam, (1958) 102-109] for the asymptotic representation of the related Bell numbers.

This task was supported by: Office of Naval Research
Contract No. NR-042-286

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ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

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A complete asymptotic development of the Stirling numbers $S(N,K)$ of the second kind is obtained by the saddle point method.

1. INTRODUCTION

Hsu [1] has given the asymptotic expansion

$$S(N,K) \sim (\frac{1}{2}K^2)^{N-K} [1 + \sum_{s=1}^t K^{-s} f_s(N-K) + O(K^{-t-1})] / (N-K)! \quad (1)$$

for Stirling numbers $S(N,K)$ of the second kind, where $f_s(N-K)$ are polynomials of argument $N-K$ and $f_s(0) = 0$. The expansion (1) is useful only for $N-K$ small, as is indicated in Section 3. We obtain a complete asymptotic expansion of $S(N,K)$ in powers of $(N+1)^{-1}$ valid for all K , using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers.

2. ASYMPTOTICS OF $S(N,K)$

A generating function for $S(N,K)$ is

$$\left(\frac{e^z - 1}{z}\right)^K = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N,K) z^{N-K}. \quad (2)$$

*This work was partially supported by the Office of Naval Research under Contract number NR 042-286 at the Naval Postgraduate School.

Hence, the Cauchy integral formula gives

$$S(N,K) = \frac{N!}{2\pi i K!} \int_C (e^z - 1)^K z^{-N-1} dz \quad (3)$$

where the contour C encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$(t-z)e^{z-t} = te^{-t}, \quad (4)$$

where $t = (N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z = u$ is on the positive real axis with $t - 1 < u < t$. There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z = u$. Since there are no other roots of (4) for $|t - z| \leq t - u$, we may apply the Lagrange inversion formula to obtain

$$u = t - \sum_{m=1}^{\infty} m^{m-1} (te^{-t})^m / m! \quad (5)$$

convergent for $t > 1$. Another form of (4) is the identity

$$K = (N+1)(1-e^{-u})/u \quad (6)$$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour C descending from $z = u$ is taken as the line $z = u + iy$, $-\infty < y < \infty$, parallel to the imaginary axis. This path has the property that the modulus of the integrand in (3) is maximal at the saddle point $z = u$. The closed

contour C is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N > 0$. The integral in (3) then becomes

$$i(e^u - 1)^K u^{-N-1} \int_{-\infty}^{\infty} \exp \psi(u+iy) dy \quad (7)$$

where

$$\psi(z) = K \ln[(e^z - 1)/(e^u - 1)] - (N+1) \ln(z/u) . \quad (8)$$

The contribution of the various parts of the $z = u + iy$ path to the integral must now be studied. As $|\exp \psi(z)| = \exp \operatorname{Re} \psi(z)$ we have to study

$$\operatorname{Re} \psi(u+iy) = K \ln[(e^{2u-2e^u \cos y+1})^{1/2}/(e^u - 1)] - (N+1) \ln(1+y^2 u^{-2})^{1/2} .$$

We shall show that we can restrict ourselves essentially to the interval $|y| < \pi$. Since

$$1 + y^2 u^{-2} \geq 1 + \pi(2y-\pi)u^{-2}$$

for $y \geq \pi$ we have

$$(e^u - 1)^K u^{-N-1} \left| \int_{\pi}^{\infty} \exp \psi(u+iy) dy \right| < u^{1-N} (e^u + 1)^K / \pi(N-1) (1+\pi^2 u^{-2})^{1/2 N-1/2}$$

which is of $O(N^{-N} e^N)$ for K small and of $O(N^{-1} e^{-0.19N})$ for K large. Since $\psi \operatorname{Re} \psi(u+iy)$ is even the part of the integral (7) for

$|y| > \pi$ tends toward zero as $N \rightarrow \infty$. We now direct our attention to the interval $|y| < \pi$ where the saddle point at $y = 0$ gives the main contribution. The Taylor expansion of $\psi(u+iy)$, convergent for $|y| < u$, is

$$\psi = -\frac{N+1}{2u}\left(\frac{1}{u} - \frac{1}{e^u-1}\right)y^2 + (N+1) \sum_{j=1}^{\infty} \frac{(iy)^{j+2}}{(j+2)!} \left(\frac{d}{dz}\right)^{j+1} \left[\frac{1-e^{-u}}{u(e^z-1)} - \frac{1}{z}\right]_{z=u} \quad (9)$$

where the identity (6) has been used. We now make the substitutions

$$w = \left[\frac{N+1}{2}\right]^{\frac{1}{2}} \left[1 - \frac{u}{e^u-1}\right]^{\frac{1}{2}} y/u \quad (10)$$

and

$$a_j = (i w u)^{j+2} \left(\frac{d}{dz}\right)^{j+1} \left[\frac{1-e^{-u}}{u(e^z-1)} - \frac{1}{z}\right]_{z=u} / (j+2)! \left[\frac{1}{2} - \frac{\frac{1}{2}u}{e^u-1}\right]^{\frac{1}{2}j+1} \quad (11)$$

to obtain

$$S(N,K) = B \int_{-\infty}^{\infty} e^{-w^2 + f[(N+1)^{-\frac{1}{2}}]} dw \quad (12)$$

where

$$B = N! (e^u-1)^K / \pi \sqrt{2(N+1)K!} u^N \sqrt{1+u/(1-\exp u)} \quad (13)$$

and

$$f[(N+1)^{-\frac{1}{2}}] = \sum_{j=1}^{\infty} a_j (N+1)^{-\frac{1}{2}j} \quad (14)$$

On using Cauchy's theorem for derivatives we find that

$$|a_j| < L \sigma^j \quad (15)$$

where

$$L = 2w^2 u^2 / (u-1)(1-e^{1-u})(1 - \frac{u}{e^u-1}) \quad (16)$$

and

$$\sigma = \sqrt{2wu} / (1 - \frac{u}{e^u-1})^{1/2} . \quad (17)$$

We now expand $\exp f[(N+1)^{-1/2}]$ in a Taylor series of the form

$$\exp f[(N+1)^{-1/2}] = \sum_{j=0}^{\infty} b_j (N+1)^{-1/2 j} \quad (18)$$

where $b_0 = 1$ and b_j are even or odd polynomials in w according to j whether it is even or odd. By a lemma of Moser and Wyman [4]

$$|b_j| \leq L \sigma^j (1+L)^{j-1} . \quad (19)$$

By (15) and (19) we see that the series (14) and (18) are convergent for suitably large N . At this point all of the reasoning given by Moser and Wyman [2] for the asymptotic expansion of the Bell numbers applies equally well to the $S(N,K)$ expansion and we conclude that

$$S(N,K) \sim B \{ \sum_{j=0}^{s-1} (N+1)^{-j} \int_{-\infty}^{\infty} b_{2j} e^{-w^2} dw + O[(N+1)^{-s}] \} . \quad (20)$$

The first two terms of the expansion (20) have been calculated to be

$$S(N, K) \sim \frac{N! (e^u - 1)^K}{\sqrt{2\pi(N+1)} K! u^N \sqrt{1+u/(1-\exp u)}} \times$$

$$\left\{ 1 + \frac{6-u^3 (e^{2u} + 4e^u + 1) (e^u - 1)^{-3}}{8(N+1) [1+u/(1-\exp u)]^2} - \frac{5[2-u^2 (e^u + 1) (e^u - 1)^{-2}]^2}{24(N+1) [1+u/(1-\exp u)]^3} \right\} . \quad (21)$$

3. NUMERICAL EXAMPLE

The following 6-significant-figure table compares the exact values of $S(100, K)$ with values computed from (20) and (1) for k ; $1 \leq k \leq 100$:

K	<u>S(100,K) Exact</u>	<u>S(100,K) 1 term of (20)</u>	<u>S(100,K) 2 terms of (20)</u>	<u>S(100,K) 4 terms of (1)</u>
1	1.00000	1.00083	1.00000	$7.74429 \cdot 10^{-176}$
2	$6.33825 \cdot 10^{29}$	$6.34348 \cdot 10^{29}$	$6.33825 \cdot 10^{29}$	$1.81186 \cdot 10^{-115}$
3	$8.58963 \cdot 10^{46}$	$8.59672 \cdot 10^{46}$	$8.58962 \cdot 10^{46}$	$3.59066 \cdot 10^{-80}$
4	$6.69558 \cdot 10^{58}$	$6.70110 \cdot 10^{58}$	$6.69557 \cdot 10^{58}$	$2.98751 \cdot 10^{-55}$
5	$6.57384 \cdot 10^{67}$	$6.57927 \cdot 10^{67}$	$6.57384 \cdot 10^{67}$	$4.46215 \cdot 10^{-36}$
6	$9.07387 \cdot 10^{74}$	$9.08136 \cdot 10^{74}$	$9.07387 \cdot 10^{74}$	$1.41738 \cdot 10^{-20}$
7	$6.41760 \cdot 10^{80}$	$6.42291 \cdot 10^{80}$	$6.41760 \cdot 10^{80}$	$1.23857 \cdot 10^{-7}$
8	$5.05211 \cdot 10^{85}$	$5.05630 \cdot 10^{85}$	$5.05210 \cdot 10^{85}$	$1.38293 \cdot 10^4$
9	$7.31910 \cdot 10^{89}$	$7.32522 \cdot 10^{89}$	$7.31910 \cdot 10^{89}$	$5.33906 \cdot 10^{13}$
10	$2.75500 \cdot 10^{93}$	$2.75733 \cdot 10^{93}$	$2.75500 \cdot 10^{93}$	$1.41107 \cdot 10^{22}$
11	$3.44958 \cdot 10^{96}$	$3.45255 \cdot 10^{96}$	$3.44958 \cdot 10^{96}$	$4.16533 \cdot 10^{29}$
12	$1.72552 \cdot 10^{99}$	$1.72704 \cdot 10^{99}$	$1.72552 \cdot 10^{99}$	$1.97468 \cdot 10^{36}$
13	$3.96430 \cdot 10^{101}$	$3.96785 \cdot 10^{101}$	$3.96430 \cdot 10^{101}$	$1.98352 \cdot 10^{42}$
14	$4.66349 \cdot 10^{103}$	$4.66773 \cdot 10^{103}$	$4.66350 \cdot 10^{103}$	$5.24140 \cdot 10^{47}$
15	$3.06221 \cdot 10^{105}$	$3.06502 \cdot 10^{105}$	$3.06221 \cdot 10^{105}$	$4.32888 \cdot 10^{52}$
16	$1.20332 \cdot 10^{107}$	$1.20443 \cdot 10^{107}$	$1.20333 \cdot 10^{107}$	$1.28480 \cdot 10^{57}$
17	$2.99575 \cdot 10^{108}$	$2.99850 \cdot 10^{108}$	$2.99576 \cdot 10^{108}$	$1.53680 \cdot 10^{61}$
18	$4.95343 \cdot 10^{109}$	$4.95793 \cdot 10^{109}$	$4.95345 \cdot 10^{109}$	$8.15008 \cdot 10^{64}$
19	$5.65920 \cdot 10^{110}$	$5.66423 \cdot 10^{110}$	$5.65922 \cdot 10^{110}$	$2.07645 \cdot 10^{68}$
20	$4.61926 \cdot 10^{111}$	$4.62326 \cdot 10^{111}$	$4.61927 \cdot 10^{111}$	$2.72088 \cdot 10^{71}$

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
21	2.77170 10 ¹¹²	2.77402 10 ¹¹²	2.77171 10 ¹¹²	1.94406 10 ⁷⁴
22	1.25295 10 ¹¹³	1.25396 10 ¹¹³	1.25295 10 ¹¹³	7.96614 10 ⁷⁶
23	4.35876 10 ¹¹³	4.36209 10 ¹¹³	4.35877 10 ¹¹³	1.95614 10 ⁷⁹
24	1.18873 10 ¹¹⁴	1.18959 10 ¹¹⁴	1.18873 10 ¹¹⁴	2.99138 10 ⁸¹
25	2.58320 10 ¹¹⁴	2.58496 10 ¹¹⁴	2.58321 10 ¹¹⁴	2.94696 10 ⁸³
26	4.53753 10 ¹¹⁴	4.54042 10 ¹¹⁴	4.53753 10 ¹¹⁴	1.92723 10 ⁸⁵
27	6.52519 10 ¹¹⁴	6.52905 10 ¹¹⁴	6.52520 10 ¹¹⁴	8.59300 10 ⁸⁶
28	7.76973 10 ¹¹⁴	7.77396 10 ¹¹⁴	7.76973 10 ¹¹⁴	2.67528 10 ⁸⁸
29	7.73865 10 ¹¹⁴	7.74251 10 ¹¹⁴	7.73864 10 ¹¹⁴	5.94163 10 ⁸⁹
30	6.50629 10 ¹¹⁴	6.50922 10 ¹¹⁴	6.50628 10 ¹¹⁴	9.59702 10 ⁹⁰
31	4.65568 10 ¹¹⁴	4.65757 10 ¹¹⁴	4.65567 10 ¹¹⁴	1.14719 10 ⁹²
32	2.85660 10 ¹¹⁴	2.85763 10 ¹¹⁴	2.85659 10 ¹¹⁴	1.03105 10 ⁹³
33	1.51311 10 ¹¹⁴	1.51358 10 ¹¹⁴	1.51310 10 ¹¹⁴	7.06841 10 ⁹³
34	6.96173 10 ¹¹³	6.96362 10 ¹¹³	6.96170 10 ¹¹³	3.74513 10 ⁹⁴
35	2.79791 10 ¹¹³	2.79856 10 ¹¹³	2.79790 10 ¹¹³	1.55216 10 ⁹⁵
36	9.87317 10 ¹¹²	9.87504 10 ¹¹²	9.87311 10 ¹¹²	5.08767 10 ⁹⁵
37	3.07351 10 ¹¹²	3.07397 10 ¹¹²	3.07349 10 ¹¹²	1.33233 10 ⁹⁶
38	8.47721 10 ¹¹¹	8.47817 10 ¹¹¹	8.47715 10 ¹¹¹	2.81367 10 ⁹⁶
39	2.07993 10 ¹¹¹	2.08009 10 ¹¹¹	2.07991 10 ¹¹¹	4.83323 10 ⁹⁶
40	4.55640 10 ¹¹⁰	4.55659 10 ¹¹⁰	4.55636 10 ¹¹⁰	6.80712 10 ⁹⁶
41	8.94238 10 ¹⁰⁹	8.94247 10 ¹⁰⁹	8.94230 10 ¹⁰⁹	7.91868 10 ⁹⁶
42	1.57729 10 ¹⁰⁹	1.57725 10 ¹⁰⁹	1.57727 10 ¹⁰⁹	7.66078 10 ⁹⁶
43	2.50762 10 ¹⁰⁸	2.50750 10 ¹⁰⁸	2.50759 10 ¹⁰⁸	6.20269 10 ⁹⁶
44	3.60314 10 ¹⁰⁷	3.60287 10 ¹⁰⁷	3.60310 10 ¹⁰⁷	4.22802 10 ⁹⁶
45	4.69092 10 ¹⁰⁶	4.69045 10 ¹⁰⁶	4.69087 10 ¹⁰⁶	2.43965 10 ⁹⁶

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
46	5.54631 10 ¹⁰⁵	5.54563 10 ¹⁰⁵	5.54625 10 ¹⁰⁵	1.19777 10 ⁹⁶
47	5.96839 10 ¹⁰⁴	5.96753 10 ¹⁰⁴	5.96832 10 ¹⁰⁴	5.02740 10 ⁹⁵
48	5.85718 10 ¹⁰³	5.85623 10 ¹⁰³	5.85711 10 ¹⁰³	1.81206 10 ⁹⁵
49	5.25178 10 ¹⁰²	5.25084 10 ¹⁰²	5.25171 10 ¹⁰²	5.63198 10 ⁹⁴
50	4.30983 10 ¹⁰¹	4.30900 10 ¹⁰¹	4.30977 10 ¹⁰¹	1.51529 10 ⁹⁴
51	3.24224 10 ¹⁰⁰	3.24157 10 ¹⁰⁰	3.24219 10 ¹⁰⁰	3.54197 10 ⁹³
52	2.23926 10 ⁹⁹	2.23877 10 ⁹⁹	2.23922 10 ⁹⁹	7.21740 10 ⁹²
53	1.42177 10 ⁹⁸	1.42146 10 ⁹⁸	1.42175 10 ⁹⁸	1.28610 10 ⁹²
54	8.30948 10 ⁹⁶	8.30758 10 ⁹⁶	8.30936 10 ⁹⁶	2.01003 10 ⁹¹
55	4.47546 10 ⁹⁵	4.47442 10 ⁹⁵	4.47539 10 ⁹⁵	2.76288 10 ⁹⁰
56	2.22373 10 ⁹⁴	2.22322 10 ⁹⁴	2.22370 10 ⁹⁴	3.34862 10 ⁸⁹
57	1.02031 10 ⁹³	1.02007 10 ⁹³	1.02029 10 ⁹³	3.58715 10 ⁸⁸
58	4.32678 10 ⁹¹	4.32581 10 ⁹¹	4.32671 10 ⁹¹	3.40392 10 ⁸⁷
59	1.69718 10 ⁹⁰	1.69682 10 ⁹⁰	1.69716 10 ⁹⁰	2.86713 10 ⁸⁶
60	6.16213 10 ⁸⁸	6.16086 10 ⁸⁸	6.16202 10 ⁸⁸	2.14775 10 ⁸⁵
61	2.07225 10 ⁸⁷	2.07185 10 ⁸⁷	2.07221 10 ⁸⁷	1.43335 10 ⁸⁴
62	6.45799 10 ⁸⁵	6.45686 10 ⁸⁵	6.45787 10 ⁸⁵	8.53612 10 ⁸²
63	1.86594 10 ⁸⁴	1.86566 10 ⁸⁴	1.86591 10 ⁸⁴	4.54310 10 ⁸¹
64	5.00048 10 ⁸²	4.99983 10 ⁸²	5.00038 10 ⁸²	2.16381 10 ⁸⁰
65	1.24327 10 ⁸¹	1.24315 10 ⁸¹	1.24325 10 ⁸¹	9.23409 10 ⁷⁸
66	2.86851 10 ⁷⁹	2.86832 10 ⁷⁹	2.86846 10 ⁷⁹	3.53479 10 ⁷⁷
67	6.14247 10 ⁷⁷	6.14228 10 ⁷⁷	6.14235 10 ⁷⁷	1.21495 10 ⁷⁶
68	1.22080 10 ⁷⁶	1.22082 10 ⁷⁶	1.22078 10 ⁷⁶	3.75283 10 ⁷⁴
69	2.25191 10 ⁷⁴	2.25204 10 ⁷⁴	2.25186 10 ⁷⁴	1.04256 10 ⁷³
70	3.85478 10 ⁷²	3.85521 10 ⁷²	3.85470 10 ⁷²	2.60661 10 ⁷¹

K	S(100,K) Exact	S(100,K) 1 term of (20)	S(100,K) 2 terms of (20)	S(100,K) 4 terms of (1)
71	6.12206 10 ⁷⁰	6.12310 10 ⁷⁰	6.12193 10 ⁷⁰	5.86843 10 ⁶⁹
72	9.01786 10 ⁶⁸	9.01999 10 ⁶⁸	9.01766 10 ⁶⁸	1.19026 10 ⁶⁸
73	1.23149 10 ⁶⁷	1.23187 10 ⁶⁷	1.23147 10 ⁶⁷	2.17566 10 ⁶⁶
74	1.55828 10 ⁶⁵	1.55889 10 ⁶⁵	1.55825 10 ⁶⁵	3.58501 10 ⁶⁴
75	1.82584 10 ⁶³	1.82671 10 ⁶³	1.82579 10 ⁶³	5.32626 10 ⁶²
76	1.97939 10 ⁶¹	1.98054 10 ⁶¹	1.97934 10 ⁶¹	7.13564 10 ⁶⁰
77	1.98357 10 ⁵⁹	1.98495 10 ⁵⁹	1.98353 10 ⁵⁹	8.62061 10 ⁵⁸
78	1.83542 10 ⁵⁷	1.83692 10 ⁵⁷	1.83538 10 ⁵⁷	9.39120 10 ⁵⁶
79	1.56619 10 ⁵⁵	1.56769 10 ⁵⁵	1.56615 10 ⁵⁵	9.22442 10 ⁵⁴
80	1.23065 10 ⁵³	1.23202 10 ⁵³	1.23062 10 ⁵³	8.16802 10 ⁵²
81	8.88952 10 ⁵⁰	8.90092 10 ⁵⁰	8.88930 10 ⁵⁰	6.51851 10 ⁵⁰
82	5.89155 10 ⁴⁸	5.90025 10 ⁴⁸	5.89140 10 ⁴⁸	4.68689 10 ⁴⁸
83	3.57456 10 ⁴⁶	3.58062 10 ⁴⁶	3.57447 10 ⁴⁶	3.03474 10 ⁴⁶
84	1.98038 10 ⁴⁴	1.98423 10 ⁴⁴	1.98032 10 ⁴⁴	1.76835 10 ⁴⁴
85	9.98901 10 ⁴¹	1.00113 10 ⁴²	9.98874 10 ⁴¹	9.26435 10 ⁴¹
86	4.57157 10 ³⁹	4.58323 10 ³⁹	4.57144 10 ³⁹	4.35788 10 ³⁹
87	1.89083 10 ³⁷	1.89636 10 ³⁷	1.89078 10 ³⁷	1.83701 10 ³⁷
88	7.03510 10 ³⁴	7.05874 10 ³⁴	7.03487 10 ³⁴	6.92073 10 ³⁴
89	2.34174 10 ³²	2.35081 10 ³²	2.34166 10 ³²	2.32155 10 ³²
90	6.92830 10 ²⁹	6.95936 10 ²⁹	6.92803 10 ²⁹	6.89924 10 ²⁹
91	1.80767 10 ²⁷	1.81711 10 ²⁷	1.80759 10 ²⁷	1.80434 10 ²⁷
92	4.11940 10 ²⁴	4.14465 10 ²⁴	4.11920 10 ²⁴	4.11648 10 ²⁴
93	8.10134 10 ²¹	8.16026 10 ²¹	8.10087 10 ²¹	8.09946 10 ²¹
94	1.35401 10 ¹⁹	1.36586 10 ¹⁹	1.35391 10 ¹⁹	1.35393 10 ¹⁹

<u>K</u>	<u>S(100,K)</u> <u>Exact</u>	<u>S(100,K)</u> <u>1 term of (20)</u>	<u>S(100,K)</u> <u>2 terms of (20)</u>	<u>S(100,K)</u> <u>4 terms of (1)</u>
95	1.88471 10 ¹⁶	1.90493 10 ¹⁶	1.88453 10 ¹⁶	1.88469 10 ¹⁶
96	2.12499 10 ¹³	2.15369 10 ¹³	2.12469 10 ¹³	2.12499 10 ¹³
97	1.86376 10 ¹⁰	1.89674 10 ¹⁰	1.86334 10 ¹⁰	1.86376 10 ¹⁰
98	1.19254 10 ⁷	1.22200 10 ⁷	1.19202 10 ⁷	1.19254 10 ⁷
99	4.95000 10 ³	5.14199 10 ³	4.94451 10 ³	4.95000 10 ³
100	1.00000	1.08086	9.94331 10 ⁻¹	1.00000

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1. ORIGINATING ACTIVITY (Corporate author)

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2a. REPORT SECURITY CLASSIFICATION

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2b. GROUP

3. REPORT TITLE

Asymptotics of Stirling Numbers of the Second Kind

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Scientific

5. AUTHOR(S) (First name, middle initial, last name)

W. E. Bleick and Peter C. C. Wang

6. REPORT DATE

20 November 1972

7a. TOTAL NO. OF PAGES

19

7b. NO. OF REFS

4

8a. CONTRACT OR GRANT NO.

NR-042-286

b. PROJECT NO.

9a. ORIGINATOR'S REPORT NUMBER(S)

NPS-53WG72113A

c.

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

d.

10. DISTRIBUTION STATEMENT

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11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Probability and Statistics Branch of
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13. ABSTRACT

A complete asymptotic development of the Stirling numbers $S(N, K)$ of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans. Roy. Soc. Canad., 49(1955) 49-54] and de Bruijn [Asymptotic Methods in Analysis, North-Holland Publishing Co., Amsterdam, (1958) 102-109] for the asymptotic representation of the related Bell numbers.

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